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A SURVEY OF FINITE-DIFFERENCE APPROXIMATIONS TO THE PRIMITIVE EQUATIONS

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ABSTRACT

The linearized primitive equations for a viscous barotropic fluid in a nonrotating frame are used to investigate the stability properties and the accuracy of several explicit finite difference approximations. Four different schemes are described and their qualities examined. For two schemes the exact numerical solution was derived and compared with the true solution of the differential system. Actual computations are performed and the errors in phase and amplitude evaluated to test the theoretical results.

SYMBOLS

$\alpha_{1,2}$	initial amplitudes
$a_{1,2} \equiv \alpha_{1,2} e^{\frac{-4\pi^2}{L^2} At}$	amplitudes of true solution
$a_{1,2}^*$	amplitudes of numerical solution
A	coefficient of lateral diffusion
$c_{1,2} = U \pm \gamma$	phase velocities of true solution
$c_{1,2}^*$	phase velocities of numerical solution
γ	phase velocity of gravity waves
G	amplification matrix
$g_{1,2}$	amplification factor
L	wavelength
λ	eigenvalue of G
U	constant speed of the basic flow
$\Delta x, \Delta t$	space, time interval
$\frac{\Delta a}{a} \equiv \frac{a - a^*}{a}$	relative error in the amplitude
$\Delta \eta \equiv \frac{2\pi}{L} t(c - c^*)$	phase lag

ABBREVIATIONS

$C_{1,2} \equiv \frac{c_{1,2} \Delta t}{\Delta x}$
$F \equiv \frac{A \Delta t}{\Delta x^2}$
$V \equiv \frac{\gamma \Delta t}{\Delta x}$
$W \equiv \frac{U \Delta t}{\Delta x}$
$\nu \equiv \sin \frac{\pi \Delta x}{L}$
$\mu \equiv \cos \frac{\pi \Delta x}{L}$
$u_j^{(n)} \equiv u\left(j \frac{\Delta x}{2}, n \Delta t\right)$
$\bar{u}_j \equiv \frac{1}{2}(u_{j+1} + u_{j-1})$
$\delta u_j \equiv (u_{j+1} - u_{j-1})$
$\delta \bar{u}_j \equiv \frac{1}{2}(u_{j+2} - u_{j-2})$
$\delta^2 u_j \equiv (u_{j+2} + u_{j-2} - 2u_j)$
$\delta^2 \bar{u}_j \equiv \frac{1}{2}(u_{j+3} + u_{j-3} - u_{j+1} - u_{j-1})$
$Im(g) \equiv$ imaginary part of g
$Re(g) \equiv$ real part of g

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1. BASIC EQUATIONS

Because of the increasing application of the primitive equations for numerically solving problems in meteorology and oceanography, it seems worth while to examine the stability properties and accuracy of various versions of finite difference approximations. As it would be too laborious to treat the complete set of primitive equations, a linearized system, in one space variable, for a barotropic atmosphere on a non-rotating earth was employed. This restriction is not too severe because the essential features are maintained when the system is extended.

The basic differential equations are the following

$$\begin{aligned}\frac{\partial u}{\partial t} &= -U \frac{\partial u}{\partial x} - \gamma \frac{\partial p}{\partial x} + A \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial p}{\partial t} &= -U \frac{\partial p}{\partial x} - \gamma \frac{\partial u}{\partial x} + A \frac{\partial^2 p}{\partial x^2}\end{aligned}\quad (1)$$

Physically u should be looked upon as being a velocity disturbance superimposed on a constant basic flow U , and p as being proportional to the depth of the fluid. γ is the phase velocity of gravity waves and A the coefficient of lateral diffusion. γ , A are constants ≥ 0 . The first term on the right hand side describes the advection of the quantities u and p due to the basic flow, the second term defines the local changes which occur due to the presence of gravity waves, and the third term illustrates the dissipation due to friction. Although it is physically incorrect to apply a viscous term to the second equation—derived from the mass conservation law—this has been done to gain symmetry. In this way, the system (1) can easily be written in characteristic form by adding and subtracting both equations. We assume the initial conditions

$$(u \pm p) = \alpha_{1,2} e^{\frac{2\pi i}{L} x}, \quad t=0 \quad (2)$$

and a periodicity condition

$$u(x, t) = u(x + L, t); \quad p(x, t) = p(x + L, t) \quad (3)$$

instead of boundary conditions. Then the solution of (1) becomes

$$(u \pm p) = \alpha_{1,2} e^{\frac{-4\pi^2}{L^2} A t} \cdot e^{\frac{2\pi i}{L} (x - c_{1,2} t)} \quad (4)$$

where $c_{1,2} = U \pm \gamma$ and $\alpha_{1,2}$ are the initial amplitudes which might be complex. Thus the solutions for u and p are built up by waves traveling with phase velocities c_1 and c_2 into the positive x -direction. The amplitudes of these waves decrease exponentially with time due to frictional dissipation. General solutions for given initial conditions can be obtained by applying Fourier expansion. Also the solutions of the finite difference approximation can be represented by Fourier series. In order to get an overall insight into the accuracy and

stability of the various finite difference schemes, it is therefore sufficient to investigate the behavior of the solutions for different Fourier terms.

2. STABILITY

Each solution of (1) has the property of giving waves, which decay proportional to $\exp(-4\pi^2 L^{-2} A t)$. In order to be called numerically stable the solution of a finite difference approximation to (1) should also consist of waves whose amplitudes do not grow with time. Substituting a typical Fourier term

$$\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix} e^{\frac{2\pi i}{L} x}$$

into the finite difference equations, the solution for the time $t = n\Delta t$, where n is the number of time steps, can be written

$$\begin{pmatrix} \hat{u}^{(n)} \\ \hat{p}^{(n)} \end{pmatrix} = G \begin{pmatrix} \hat{u}^{(n-1)} \\ \hat{p}^{(n-1)} \end{pmatrix} = G^n \begin{pmatrix} \hat{u}^{(0)} \\ \hat{p}^{(0)} \end{pmatrix}$$

$\begin{pmatrix} \hat{u}^{(0)} \\ \hat{p}^{(0)} \end{pmatrix}$ is a constant vector describing the initial amplitude. G is called the amplification matrix; the elements of G depend on $c_{1,2}$, A , L , Δx , and Δt , where Δx is the grid interval. To get a stable numerical solution as stated above, G has to be bounded. This leads to the von Neumann necessary condition for stability, namely, that the eigenvalues $\lambda_{1,2}$ of G are not allowed to exceed 1 in absolute value (Lax and Richtmyer [4]). Thus:

$$|\lambda_{1,2}| \leq 1 \text{ for all possible } L \quad (5)$$

This condition mostly implies a specific choice of the parameters Δx and Δt . In the following, several explicit finite difference approximations to (1) are to be examined. Disregarding the friction term that has been approximated only to first order accuracy, all schemes but the first (the scheme Ia) possess second-order accuracy. Staggered grids were used since they allow a better approximation of space derivatives than the non-staggered grids. In the difference schemes Ia and Ib, u and p are prescribed at different grid points and different time levels. The form of the schemes II and III suggested another type of staggering prescribing u and p at the same grid points but shifting these points half a grid interval each time step.

3. SCHEME I

The first finite difference approximation to (1) to be treated has the following form

$$\begin{aligned}u_j^{(n+1)} &= u_j^{(n)} - [W\delta\bar{u}_j^{(n)} - \frac{1}{2}W^2\delta^2 u_j^{(n)}] - V\delta p_j^{(n+1/2)} + F\delta^2 u_j^{(n)} \\ p_j^{(n+3/2)} &= p_j^{(n+1/2)} - [W\delta\bar{p}_j^{(n+1/2)} - \frac{1}{2}W^2\delta^2 p_j^{(n+1/2)}] \\ &\quad - V\delta u_j^{(n+1)} + F\delta^2 p_j^{(n+1/2)} \quad (\text{Ia})\end{aligned}$$

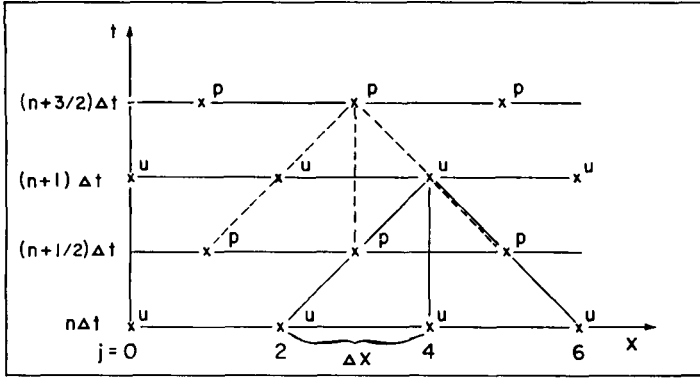


FIGURE 1.—Arrangement of grid points in the x, t -plane and values needed to compute $u_4^{(n+1)}$ and $p_3^{(n+3/2)}$.

It uses the following grid with respect to x and t (fig. 1):

u -values are prescribed at even j and integer times, p -values are prescribed at odd j and half-odd-integer times. The principal form of the finite difference approximation Ia was suggested by C. Leith (personal communication, 1963) for numerical experiments of the general circulation of the atmosphere. Note that the advection terms $U(\partial u/\partial x)$, $U(\partial p/\partial x)$ have been approximated through quadratic interpolation between three grid points by those terms standing inside the brackets. This interpolation is not quite complete as will be outlined later. When starting with known initial values $u^{(0)}$, $p^{(0)}$, the above scheme requires an extrapolation in time in order to get the value $p^{(1/2)}$; this can be done by applying a half time step procedure initially.

In actual computations practically the same results were obtained with a slightly different scheme in which $p^{(n+3/2)}$ and $p^{(n+1/2)}$ were replaced by $p^{(n+1)}$ and $p^{(n)}$; this system avoids the initial extrapolation. Both schemes possess the same stability properties. Substituting a Fourier term into (Ia) the solution can be written

$$\begin{pmatrix} \hat{u}^{(n)} \\ \hat{p}^{(n+1/2)} \end{pmatrix} = G^n \begin{pmatrix} \hat{u}^{(0)} \\ \hat{p}^{(1/2)} \end{pmatrix}$$

The eigenvalues of G become

$$\lambda_{1,2} = 1 - (4F + 2W^2 + 2V^2)\nu^2 - 2i\mu\nu W \pm 2iV\nu[1 - (4F + 2W^2 + V^2)\nu^2 - 2i\mu\nu W]^{1/2}. \quad (6)$$

The necessary condition for numerical stability is that $|\lambda_{1,2}|$ does not exceed one as was outlined in section 2.

If $\frac{U\Delta t}{\Delta x} \equiv W$ would be zero, the absolute value of $\lambda_{1,2}$ would be $|\lambda_{1,2}| = 1 - 4F\nu^2 \leq 1$ provided $4F + V^2 \leq 1$. The same is true if $\gamma\Delta t/\Delta x \equiv V$ vanishes, then one would obtain $|\lambda_{1,2}|^2 = (1 - 4F\nu^2)^2 - 4W^2\nu^4(1 - W^2 - 4F) \leq 1$ if $2F + W^2 \leq 1$. Thus if either U or γ is zero, the finite difference scheme Ia is stable under appropriate conditions

prescribing a specific relationship between the parameters Δt and Δx . As it is difficult to give a closed expression for $|\lambda_{1,2}|$ with respect to all possible wavelengths if both W and V are different from zero, we will be satisfied to evaluate $|\lambda_{1,2}|$ for long waves only, i.e., for $\nu \equiv \sin \frac{\pi\Delta x}{L} \ll 1$. We assume also that $F\nu^2 \ll 1$, $V\nu \ll 1$, $W\nu \ll 1$, and obtain

$$|\lambda_{1,2}|^2 \approx 1 - 4\left(\frac{\pi\Delta x}{L}\right)^2 (2F \mp VW) \quad (7)$$

It is immediately seen that our scheme Ia gives exponentially growing solutions with time if friction is neglected ($F=0$) since then $|\lambda_1| > 1$ for $W > 0$, i.e., $U > 0$. If $F \neq 0$, stability is only retained if $2A > |U|\gamma\Delta t$. In order to satisfy this condition A has to be rather large ($> 10^{10}$ cm.² sec.⁻¹) to allow for a reasonable timestep (> 200 sec.) if, simulating atmospheric conditions, $|U|\gamma$ is in the order of 10^8 cm.² sec.⁻². Practical computations, the results of which will be discussed in section 6, proved this fact. There, instability also occurred for nearly all shorter wave components when the above condition was violated and was confined mainly to the wave with phase velocity $|U| + \gamma$. Also, for some short waves the wave with phase velocity $|U| - \gamma$ showed slight instability. An exception to this is the shortest wave $L = 2\Delta x$, which is stable when $2F + W^2 + V^2 \leq 1$.

A finite difference approximation to (1) that becomes unstable when friction is omitted or too small is not acceptable. We shall therefore re-examine our scheme with the goal of eliminating the unfavorable stability condition and arrive eventually at a stable scheme independent of the magnitude of A .

Evidently the terms in the brackets of (Ia) should represent the advection of u and p at the corresponding centered time levels which shall be written $W\delta\tilde{u}_j^{(n+1/2)}$ and $W\delta\tilde{p}_j^{(n+1)}$ respectively. Since u at time $(n+\frac{1}{2})\Delta t$ and p at time $(n+1)\Delta t$ are not given directly by the grid (see fig. 1) the fictitious values have been denoted by $\tilde{u}_j^{(n+1/2)}$ and $\tilde{p}_j^{(n+1)}$. It can be shown that a stable system according to (5) is obtained if $\delta\tilde{u}_j^{(n+1/2)}$ —and similarly $\delta\tilde{p}_j^{(n+1)}$ —is approximated by the grid point values $\frac{1}{2}(\delta\tilde{u}_j^{(n+1)} + \delta u_j^{(n)})$. This leads, however, to a partly implicit scheme which is inconvenient to handle; it shows, on the other hand, that the shortcomings of the scheme Ia must be due to an incorrect approximation of the advection terms with respect to the centered time. To avoid an implicit scheme and yet arrive at a sufficiently correct approximation we follow a procedure suggested by Lax and Wendroff [5] and expand $\tilde{u}_j^{(n+1/2)}$ and $\tilde{p}_j^{(n+1)}$ into Taylor series in time retaining only those terms which guarantee the desired accuracy. This leads, for example, to $\tilde{u}_j^{(n+1/2)} = \tilde{u}_j^{(n)} + \left\langle \frac{\partial u}{\partial t} \right\rangle_j^{(n)} \frac{\Delta t}{2}$ where $\left\langle \frac{\partial u}{\partial t} \right\rangle_j^{(n)}$ is a finite difference approximation to $\partial u/\partial t$ in (1) which shall be expressed in the following way:

$$\frac{\Delta t}{2} \left\langle \frac{\partial u}{\partial t} \right\rangle_j^{(n)} = -\frac{1}{2} W \delta u_j^{(n)} - \frac{1}{2} V \delta \bar{p}_j^{(n+1/2)} + \frac{1}{2} F \delta^2 \bar{u}_j^{(n)}.$$

The second term on the right-hand side is not evaluated at the correct time level $n\Delta t$, since p is not defined there, but this causes only an error of third order in Δt .

The overall scheme can now be written as follows:

$$\tilde{u}_j^{(n+1/2)} = \bar{u}_j^{(n)} - \frac{1}{2} W \delta u_j^{(n)} - \left[\frac{1}{2} V \delta \bar{p}_j^{(n+1/2)} - \frac{1}{2} F \delta^2 \bar{u}_j^{(n)} \right]$$

$$u_j^{(n+1)} = u_j^{(n)} - W \delta \tilde{u}_j^{(n+1/2)} - V \delta p_j^{(n+1/2)} + F \delta^2 u_j^{(n)}$$

$$\tilde{p}_j^{(n+1)} = \bar{p}_j^{(n+1/2)} - \frac{1}{2} W \delta p_j^{(n+1/2)} - \left[\frac{1}{2} V \delta \bar{u}_j^{(n+1)} - \frac{1}{2} F \delta^2 \bar{p}_j^{(n+1/2)} \right]$$

$$p_j^{(n+3/2)} = p_j^{(n+1/2)} - W \delta \tilde{p}_j^{(n+1)} - V \delta u_j^{(n+1)} + F \delta^2 p_j^{(n+1/2)}. \quad (\text{Ib})$$

This scheme has second order accuracy in Δt and Δx except for the friction terms which have first order accuracy in Δt and second order accuracy in Δx . Referring to the x, t -diagram (fig. 1), $\tilde{u}_j^{(n+1/2)}$ and $\tilde{p}_j^{(n+1)}$ are to be defined at the same points where $p_j^{(n+1/2)}$ and $u_j^{(n+1)}$ are given respectively. A very similar scheme was described by Richtmyer [11] but its stability condition was not determined. Another investigation of this system by A. Kasahara [2] can be found in this issue of the *Monthly Weather Review*.

The scheme Ib has been written as a two-step scheme, where $\tilde{u}^{(n+1/2)}$, $u^{(n+1)}$, $\tilde{p}^{(n+1)}$, $p^{(n+3/2)}$ have to be evaluated successively. This way of representation might be convenient if the same method is to be adopted for more complicated systems. We can, however, easily arrive at a one-step scheme by eliminating \tilde{u} and \tilde{p} in the second and fourth equations. In doing so for the second equation, for example, we obtain:

$$u_j^{(n+1)} = u_j^{(n)} - W \delta \bar{u}_j^{(n)} + \frac{1}{2} W^2 \delta^2 u_j^{(n)} + \frac{1}{2} V W \delta^2 \bar{p}_j^{(n+1/2)} - \frac{1}{2} W F \delta^3 \bar{u}_j^{(n)} - V \delta p_j^{(n+1/2)} + F \delta^2 u_j^{(n)}.$$

Compared with the corresponding expression in the original scheme, Ia, an additional second order term $\frac{1}{2} V W \delta^2 \bar{p}_j^{(n+1/2)}$ appears. Both schemes become identical only if the terms in the brackets in the first and third equations of (Ib) are omitted. Note that from the fields $u^{(n)}$, $p^{(n+1/2)}$ the field $u^{(n+1)}$ can be computed and then from the fields $u^{(n+1)}$, $p^{(n+1/2)}$ the field $p^{(n+3/2)}$.

As it is still very difficult to give a closed expression of $|\lambda|$ for all wavelengths we shall again confine the computations to the long wave components. This gives:

$$|\lambda_{1,2}|^2 \approx \left(1 - 4F \frac{\pi^2 \Delta x^2}{L^2} \right)^2 - 4 \left(\frac{\pi \Delta x}{L} \right)^4 W^2 \left[1 - W^2 \pm \frac{V}{W} (W^2 + 2F) \right]. \quad (8a)$$

If F is neglected in the first and third equations, this yields:

$$|\lambda_{1,2}|^2 \approx \left(1 - 4F \frac{\pi^2 \Delta x^2}{L^2} \right)^2 - 4 \left(\frac{\pi \Delta x}{L} \right)^4 W^2 \left[1 - (W^2 + 4F) \left(1 \mp \frac{V}{W} \right) \right]. \quad (8b)$$

In either case, it is now possible, without restrictive assumptions, to attain stability at least for long waves. If we neglect friction ($F=0$), and then $|\lambda_{1,2}|$ does not exceed 1 when $W^2 \pm VW \leq 1$, thus requiring that

$$(U^2 + |U|\gamma) \left(\frac{\Delta t}{\Delta x} \right)^2 \leq 1$$

which is always possible to achieve. Including the friction will, for proper values of F , even tend to improve the stability. So far we have no proof, however, that our scheme is stable or if the stability requirement above is valid for all wavelengths. Numerical computations of the eigenvalues for $F=0$, performed by Dr. Kasahara, NCAR, revealed however that the scheme Ib is stable for all wave components and that the shortest possible wavelength $L=2\Delta x$ imposes the most restrictive condition for stability, requiring

$$\left(\frac{U\Delta t}{\Delta x} \right)^2 + \frac{\gamma\Delta t}{\Delta x} \leq 1.$$

(For this wave the eigenvalues are identical to (6) with $\nu=1$ and $\mu=0$.)* Thus we can state that for vanishing F the scheme is stable if this condition is met. Including the friction will probably change the stability requirement into:

$$2 \frac{A\Delta t}{\Delta x^2} + \left(\frac{U\Delta t}{\Delta x} \right)^2 + \frac{\gamma\Delta t}{\Delta x} \leq 1.$$

Examples of an actual computation with the schemes Ia and Ib will be given later in section 6.

It should be mentioned, however, that scheme Ib, modified for a nonstaggered grid in space, with both $u^{(n)}$ and $p^{(n+1/2)}$ prescribed at, say, even j , might not be stable in the same range. In this case, applying the same principles as before, it seems reasonable to transform the terms in the second equation of (Ib), for example, su that:

$$u_j^{(n+1)} = u_j^{(n)} - W \delta \bar{u}_j^{(n)} + \frac{1}{2} W^2 \delta^2 u_j^{(n)} + \frac{1}{2} V W \delta^2 p_j^{(n+1/2)} - V \delta \bar{p}_j^{(n+1/2)}.$$

For simplicity, F has been neglected. Assuming again $\nu < 1$, the eigenvalues for this system become:

$$|\lambda_{1,2}|^2 \approx 1 - 4 \left(\frac{\pi \Delta x}{L} \right)^4 [(W^2 \mp VW)(1 - W^2)] \quad (9)$$

and stability for long waves is obtained only if $V < |W| < 1$, that is, for supersonic flow. The same result was derived by Kasahara [2] through numerical computation of the eigenvalues.

*It would seem from an inspection of this equation that the stability condition should read $2 \left(\frac{U\Delta t}{\Delta x} \right)^2 + \left(\frac{\gamma\Delta t}{\Delta x} \right)^2 \leq 1$ because this results, as can immediately be verified, in $|\lambda_{1,2}| = 1 - 2 \left(\frac{U\Delta t}{\Delta x} \right)^2 < 1$. But this condition is more limiting than that above.

4. SCHEME II (TWO-STEP LAX-WENDROFF SCHEME)

Scheme II consists, similar to Ib, of two different steps and is called according to Richtmyer [12] the two-step Lax-Wendroff scheme [5]. In its staggered form it is identical to the one-dimensional version of the scheme proposed by Phillips [9]. It is also closely related to the system developed by Knox [3]. The finite difference equations take the following form,

$$\begin{aligned} u_j^{(n+1/2)} &= \bar{u}_j^{(n)} - \frac{1}{2}W\delta u_j^{(n)} - \frac{1}{2}V\delta p_j^{(n)} + \frac{1}{2}F\delta^2 \bar{u}_j^{(n)} \\ p_j^{(n+1/2)} &= \bar{p}_j^{(n)} - \frac{1}{2}W\delta p_j^{(n)} - \frac{1}{2}V\delta u_j^{(n)} + \frac{1}{2}F\delta^2 \bar{p}_j^{(n)} \\ u_j^{(n+1)} &= u_j^{(n)} - W\delta u_j^{(n+1/2)} - V\delta p_j^{(n+1/2)} + F\delta^2 u_j^{(n)} \\ p_j^{(n+1)} &= p_j^{(n)} - W\delta p_j^{(n+1/2)} - V\delta u_j^{(n+1/2)} + F\delta^2 p_j^{(n)}. \quad (\text{II}) \end{aligned}$$

u and p are given at the same grid points and are arranged according to the x, t -diagram shown by figure 2.

At half numbers of time steps, the grid points have odd j ; at whole numbers of time steps j is even. In the second step consisting of the two last equations of (II) the variables on the right-hand side are centered in time except for the frictional term and are approximated by the first step in a similar way as in (Ib). In the second step, the frictional term is not centered in time to avoid an interpolation between four grid points, as in the first step.

The exact solution can be obtained quite easily for this scheme, as u and p are prescribed at the same grid points. By adding and subtracting the equations for u and p one obtains equations in the variables $(u \pm p)$. Applying a Fourier term, the solution after n time steps becomes:

$$(u_j^{(n)} \pm p_j^{(n)}) = g_{1,2}^n \alpha_{1,2} e^{\frac{2\pi i}{L} x} = |g_{1,2}|^n \alpha_{1,2} e^{-\frac{2\pi i}{L} n \Delta t c_{1,2}^*} e^{\frac{2\pi i}{L} x}$$

where

$$g_{1,2} = 1 - 2(2F + C_{1,2}^2)\nu^2 - 2i\mu\nu C_{1,2}(1 - 2F\nu^2)$$

$$|g_{1,2}|^2 = (1 - 4F\nu^2)^2 - 4C_{1,2}^2\nu^4[1 - C_{1,2}^2 - 4F\nu^2(1 - F\mu^2)]$$

$$-\frac{2\pi}{L} \Delta t c_{1,2}^* = \arctg \frac{Im(g_{1,2})}{Re(g_{1,2})}. \quad (10)$$

Here the amplification matrix reduces to the amplification factor $g_{1,2}$, which is identical to its eigenvalue $\lambda_{1,2}$. Since $|g_{1,2}| \alpha_{1,2} = \alpha_{1,2}^*$ is the amplitude of the numerical solution which is not allowed to grow with time, a necessary and sufficient condition for stability is that $|g_{1,2}| = |\lambda_{1,2}| \leq 1$ according to (5). This stability requirement is fulfilled for each possible L when

$$2 \frac{A\Delta t}{\Delta x^2} + (|U| + \gamma)^2 \left(\frac{\Delta t}{\Delta x} \right)^2 \leq 1 \quad (11)$$

If $F=0$, i.e., if the natural decay of waves is neglected, it is seen from (10) that there remains an artificial

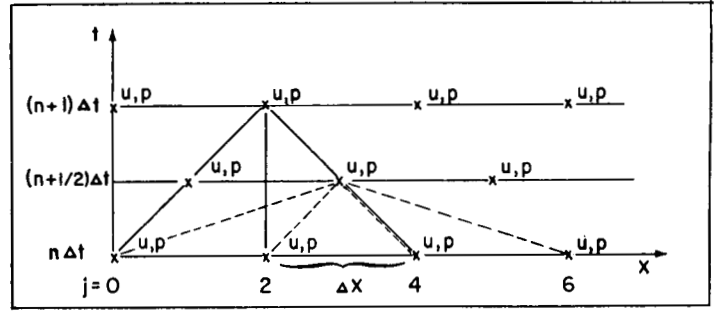


FIGURE 2.—Arrangement of grid points in the x, t -plane and values needed to compute $u_3^{(n+1/2)}$, $p_3^{(n+1/2)}$ and $u_2^{(n+1)}$, $p_2^{(n+1)}$.

damping proportional to $C_{1,2}^2\nu^4$ and $0 \leq 1 - C_{1,2}^2 < 1$. This damping affects mainly the shorter waves. To investigate the influence of this artificial damping on the larger wave components, we assume $\nu \ll 1$. Then after n time steps the relative error in the amplitude $\frac{a-a^*}{a} \equiv \left(\frac{\Delta a}{a} \right)$ is:

$$\left(\frac{\Delta a}{a} \right)_{1,2} \approx 2n \left(\frac{\pi \Delta x}{L} \right)^4 \left[-2F \left(\frac{1}{3} - 2F \right) + C_{1,2}^2(1 - C_{1,2}^2) \right] \quad (12)$$

This expression holds as long as $n \left(\frac{\pi \Delta x}{L} \right)^4 \ll 1$. If we

again put $F=0$, then $C_{1,2}^2=1/2$ would give the largest amount of artificial damping for the wave with phase velocity $c_{1,2}$. With this value, a relative error in the amplitude of 10 percent is reached after 20 time steps for $L/\Delta x=10$, after 320 time steps for $L/\Delta x=20$, and after 1600 time steps for $L/\Delta x=30$. Thus the artificial damping decreases quite rapidly—proportional to $(\Delta x/L)^4$ —with increasing wavelength.

Evidently either increasing or decreasing the previously applied value of $C_{1,2}^2=1/2$ would result in a smaller error in the amplitude for the same number of time steps. To be able to compare the error at a fixed time rather than after a fixed number of time steps, we assume that $C_{1,2} \equiv \frac{c_{1,2}\Delta t}{\Delta x}$ is changed by Δt only and that Δx and $c_{1,2}$ are kept constant. For simplicity the frictional term F is ignored.

If, as reference time, $t_0 = \tilde{t}_0$ is chosen, where \tilde{t}_0 is the maximum time step permitted to retain stability according to (11) (if $A=0$) and the actual time step is taken as $\Delta t = \rho \tilde{\Delta t}$ where $0 < \rho \leq 1$ ($1/\rho$ is the number of time steps needed to reach the time t_0), then the relative error at time t_0 becomes:

$$\begin{aligned} \left(\frac{\Delta a}{a} \right)_{1,2} &= 1 - [1 - 4\rho^2 \tilde{C}_{1,2}^2 \nu^4 (1 - \rho^2 \tilde{C}_{1,2}^2)]^{1/2\rho} \\ &\approx 2\rho \tilde{C}_{1,2}^2 \left(\frac{\pi \Delta x}{L} \right)^4 (1 - \rho^2 \tilde{C}_{1,2}^2) \end{aligned} \quad (13)$$

where

$$\tilde{C}_{1,2} = \frac{c_{1,2}\tilde{\Delta t}}{\Delta x} \left(\text{if } U \geq 0 \text{ then } \tilde{C}_1^2 = 1, \right. \\ \left. \tilde{C}_2^2 = 1 - 4U\gamma \left(\frac{\tilde{\Delta t}}{\Delta x} \right)^2, \text{ and } \tilde{\Delta t}^2 = \frac{\Delta x^2}{c_1^2} \right).$$

Here the largest error is obtained if $\tilde{C}_{1,2}^2 \rho^2 = 1/3$. Furthermore it is seen that for a fixed time $\left(\frac{\Delta a}{a} \right)_{1,2} \rightarrow 0$ if $\Delta t \rightarrow 0$ and $\Delta x = \text{constant}$.

To investigate the phase differences between the numerical and true solution, we again assume $\nu \ll 1$. Then

$$\frac{2\pi}{L} n \Delta t c_{1,2}^* \approx \frac{2\pi}{L} n \Delta t c_{1,2} \left[1 - \frac{2}{3} \left(\frac{\pi \Delta x}{L} \right)^2 (1 - C_{1,2}^2 - 3F) \right]$$

and the phase lag $\Delta \eta \equiv \frac{2\pi}{L} n \Delta t (|c| - |c^*|)$ after n time steps becomes:

$$\Delta \eta \approx \frac{4}{3} \left(\frac{\pi \Delta x}{L} \right)^3 n |C_{1,2}| (1 - C_{1,2}^2 - 3F). \quad (14)$$

Thus the phase velocity of the numerical solution is smaller in absolute value than that of the true solution when $C_{1,2}^2 + 3F < 1$ and depends on the wavelength. Opposite to the true solution (4), the numerical solution creates dispersive waves. If we neglect F and again assume that only the time step is diminished by $\Delta t = \rho \tilde{\Delta t}$, we obtain a phase lag at the time $t_0 = \tilde{\Delta t}$ of

$$(\Delta \eta)_{1,2} \approx \frac{4}{3} \left(\frac{\pi \Delta x}{L} \right)^3 |\tilde{C}_{1,2}| (1 - \rho^2 \tilde{C}_{1,2}^2). \quad (15)$$

Even if the error in the amplitude becomes small for $\rho \rightarrow 0$, i.e., $\Delta t \rightarrow 0$ according to (13), the phase error approaches its maximum value. If, for example, $U \geq 0$, then it follows

from (11) (with $A=0$) that $\tilde{C}_1^2 = (U + \gamma)^2 \frac{\tilde{\Delta t}^2}{\Delta x^2} = 1$. If $\Delta t/\Delta x$ is chosen in such a way that $|C_1| = 1/2$, i.e., $\rho = 1/2$, then for $L = 10\Delta x$ the relative error in amplitude at $t_0 = \Delta t$ would be 0.75 percent and the phase lag would be 1.8° . Diminishing the time step by one half gives $\rho = 1/4$ and the errors for the time t_0 then become 0.41 percent and 2.2° , respectively. Obviously the errors in both amplitude and phase would vanish if $C_{1,2}^2 = \tilde{C}_{1,2}^2 = 1$; but this condition cannot be satisfied for both waves if $U \neq 0$.

To refine only the time interval may reduce the errors in amplitude. This leads, however, to increasing errors in phase. If both F and $C_{1,2}$ are different from zero, the best way to diminish the errors would be to refine both the time and grid interval such that $(\Delta t/\Delta x^2)(A + \sqrt{c_{1,2}^2 \Delta x^2 + A^2}) = \text{const.} \leq 1$. This expression is derived from solving (11) with respect to Δt . As long as this relation is met when Δt and Δx are made smaller and smaller, the stability requirement is automatically fulfilled

and the true solution is approached in the limit $\Delta t \rightarrow 0$, $\Delta x \rightarrow 0$. It is seen that as long as $c_{1,2} \Delta x^2 \gg A^2$, Δt and Δx should be refined at the same rate, as in the case of pure hyperbolic systems. When $c_{1,2}^2 \Delta x^2$ becomes smaller than A^2 , Δt should be made finer and finer proportional to Δx^2 , as in the case of pure parabolic systems.

We have seen that a wavelength represented by 20 grid intervals can be damped artificially by 10 percent after 320 time steps. If we, simulating conditions in numerical weather forecasting, take $\Delta x = 300$ km., $c_1 = 300$ m. sec.⁻¹ and $C_1^2 = 1/2$, then Δt becomes 700 sec. With $L = 20\Delta x = 6000$ km., this wave will be damped artificially by 10 percent after 2.6 days and will have a phase lag of 33° (F is assumed to be zero).

It would have been more desirable if this damping were confined to the very short waves ($L < 10\Delta x$). One could perhaps think that applying the second time centered step of (II) several times, say N times, before returning to the first step would reduce the artificial damping. If $N=3$, for example, we compute $u^{(1/2)}$, $p^{(1/2)}$ from the first step and insert these values into the second step to obtain $u^{(1)}$, $p^{(1)}$. Now we continue to employ the equations for the second step also to get the values of $u^{(3/2)}$, $p^{(3/2)}$ and $u^{(2)}$, $p^{(2)}$ and then go back to the equations for the first step to evaluate $u^{(5/2)}$, $p^{(5/2)}$. Repeating this procedure leads to the following errors in amplitude at time $t = n\Delta t$ ($F=0$).

For $N=3$

$$\left(\frac{\Delta a}{a} \right)_{1,2} = 1 - [1 - 16C_{1,2}^2 \nu^4 (1 - C_{1,2}^2) (1 - 2C_{1,2}^2 \nu^2)^2]^{n/4}.$$

For $N=5$

$$\left(\frac{\Delta a}{a} \right)_{1,2} = 1 - \left[1 - 36C_{1,2}^2 \nu^4 (1 - C_{1,2}^2) (1 - 4C_{1,2}^2 \nu^2)^2 \right. \\ \left. \left(1 - \frac{4}{3} C_{1,2}^2 \nu^2 \right)^2 \right]^{n/6}.$$

Whereas the original scheme ($N=1$) yields, according to (10):

$$\left(\frac{\Delta a}{a} \right)_{1,2} = 1 - [1 - 4C_{1,2}^2 \nu^4 (1 - C_{1,2}^2)]^{n/2}$$

If we assume $C_{1,2}^2 < 1$, the schemes with $N=3$ and $N=5$ show a smaller damping than the original scheme ($N=1$) only for certain short waves, whereas the damping is larger for the very long waves. Apparently for increasing N and $F=0$ there occur more and more places where $(\Delta a/a)_{1,2}$ becomes very small or zero beginning at short wavelengths, i.e., relatively large values of $C_{1,2}^2 \nu^2 < 1$ and extending gradually to longer waves so that for $N \rightarrow \infty$, $(\Delta a/a)_{1,2} \rightarrow 0$ for all wave numbers. Thus, if N is small, there is no improvement from applying this method because the advantage of getting rid of the short waves—we shall explain later why this might be desirable—is lost and the disadvantage of damping the longer waves is maintained.

If $N \rightarrow \infty$, which means in practice that except initially only the two last equations of (II) are used, the scheme obtained is called the leapfrog scheme; it will be investigated in the next section.

5. SCHEME III (LEAPFROG SCHEME)

Taking the two last equations of (II) for all times ($n=0, 1/2, 1, 3/2, \dots$), except for the starting point where we apply the first two equations to get the values at $t=\frac{1}{2}\Delta t$ leads to the so-called leapfrog scheme. This scheme is commonly used in numerical computations (see Miyakoda [7]).

After a certain number of time steps when the influence of the initial guess has practically disappeared, the numerical solution becomes

$$(u_j^{(n)} \pm p_j^{(n)}) = g_{1,2}^n \alpha_{1,2} e^{\frac{2\pi i}{L} x} = |g_{1,2}|^n \alpha_{1,2} e^{-\frac{2\pi i}{L} n \Delta t c_{1,2}} e^{\frac{2\pi i}{L} x} \quad (16)$$

where

$$g_{1,2} = 1 - 4F\nu^2 - 2iC_{1,2}\nu\sqrt{1 - (4F + C_{1,2}^2)\nu^2}$$

and

$$-\frac{2\pi}{L} \Delta t c_{1,2} = \arctg \frac{Im(g_{1,2})}{Re(g_{1,2})}$$

The stability condition is fulfilled when

$$\frac{4A\Delta t}{\Delta x^2} + (|U| + \gamma)^2 \left(\frac{\Delta t}{\Delta x}\right)^2 \leq 1. \quad (17)$$

Then the amplification factor has the absolute value

$$|g_{1,2}| = 1 - 4F\nu^2 \leq 1. \quad (18)$$

Thus if the natural decay of waves is neglected, it is seen that the leapfrog scheme contains no artificial damping.

For long waves $\nu \ll 1$ the relative error in the amplitude becomes

$$\left(\frac{\Delta a}{a}\right)_{1,2} \approx -\frac{4}{3}n \left(\frac{\pi \Delta x}{L}\right)^4 F(1-6F) \quad (19)$$

and the phase lag is given by

$$(\Delta \eta)_{1,2} \approx \frac{1}{3} \left(\frac{\pi \Delta x}{L}\right)^3 n |C_{1,2}| (1 - C_{1,2}^2 - 12F). \quad (20)$$

Compared with the corresponding results of (12) and (14) for the Lax-Wendroff scheme, the latter has about four times larger phase errors; also, the errors in the amplitude are larger.

6. RESULTS OF NUMERICAL COMPUTATIONS

Practical computations were performed with all schemes described in the previous chapters. In agreement with (2), the initial conditions $u(x,0) = \cos \frac{2\pi}{L}x$, $p(x,0) = 0$ were applied; i.e., $\alpha_1 = \alpha_2 = 1$. The periodicity conditions

(3) were extended to $\frac{\partial^k u}{\partial x^k}(0,t) = \frac{\partial^k u}{\partial x^k}(L,t)$, $\frac{\partial^k p}{\partial x^k}(0,t) = \frac{\partial^k p}{\partial x^k}(L,t)$ ($k=0, 1, 2, \dots$) in order to get also those values of u and p for $x < 0$, $x > L$ to which the numerical schemes refer. The exact solution (4) also satisfies these boundary conditions. $u_j^{(n)}$ and $p_j^{(n)}$ were computed from the finite difference scheme and then $u_j^{(n)} \pm p_j^{(n)}$ was formed and compared with the corresponding true solution which is simply $\cos \frac{2\pi}{L} c_{1,2} t e^{-\frac{4\pi^2}{L^2} A t}$. Since in the schemes Ia and Ib u and p are not given at the same grid points and at the same time, a linear interpolation in both time and space was applied to get the desired value $p_j^{(n)}$. Since one goal of this paper is to simulate conditions in numerical weather prediction, the following parameters were presumed: $\Delta t = 4 \times 10^2 (2 \times 10^2)$ sec., $\Delta x = 2 \times 10^7$ cm., $U = 5 \times 10^3$ cm., sec.⁻¹, $\gamma = 3 \times 10^4$ cm., sec.⁻¹, $A = 10^9$ cm.², sec.⁻¹, which yields $\gamma \Delta t / \Delta x \equiv V = 0.6$ (0.3), $U \Delta t / \Delta x \equiv W = 0.1$ (0.05), $A \Delta t / \Delta x^2 \equiv F = 0.001$ (0.0005). The values in the parentheses refer to the scheme Ia. The errors at time $t = 4 \times 10^4$ sec. for different wavelengths are shown in tables 1 and 2.

These results confirm what was derived theoretically before. They show that for the wave with phase velocity c_1 the numerical amplitude in scheme Ia is larger than the true amplitude and also exceeds the initial amplitude $\alpha_1 = 1$, since $|\lambda_1|$ in (7) is larger than one ($|\lambda_1| \approx 1 + 2.8 \left(\frac{\Delta x}{L}\right)^2$); for $L = 4\Delta x$ (not presented in table 1) also, the wave with phase velocity c_2 becomes slightly unstable. The scheme Ib, however, is stable for all wavelengths and contains a

TABLE 1.—Errors in amplitude $\left(\frac{\Delta a}{a}\right)_1$ and phase $(\Delta \eta)_1$ at $t = 4 \times 10^4$ sec. for different wavelengths L .

Scheme	$L = 6\Delta x$		$L = 10\Delta x$		$L = 20\Delta x$		$L = 30\Delta x$	
	$\left(\frac{\Delta a}{a}\right)_1$	$(\Delta \eta)_1$	$\left(\frac{\Delta a}{a}\right)_1$	$(\Delta \eta)_1$	$\left(\frac{\Delta a}{a}\right)_1$	$(\Delta \eta)_1$	$\left(\frac{\Delta a}{a}\right)_1$	$(\Delta \eta)_1$
Ia.....	-2.60	250°	-0.70	60°	-0.15	8.4°	-0.07	2.8°
Ib.....	0.10	210°	0.03	40°	0.004	5.5°	0.00	1.6°
II.....	0.96	320°	0.36	80°	0.03	10.6°	0.005	3.0°
III.....	0.00	100°	0.00	22°	0.00	2.6°	0.00	0.75°

TABLE 2.—Errors in amplitude $\left(\frac{\Delta a}{a}\right)_2$ and phase $(\Delta \eta)_2$ at $t = 4 \times 10^4$ sec. for different wavelengths L .

Scheme	$L = 6\Delta x$		$L = 10\Delta x$		$L = 20\Delta x$		$L = 30\Delta x$	
	$\left(\frac{\Delta a}{a}\right)_2$	$(\Delta \eta)_2$	$\left(\frac{\Delta a}{a}\right)_2$	$(\Delta \eta)_2$	$\left(\frac{\Delta a}{a}\right)_2$	$(\Delta \eta)_2$	$\left(\frac{\Delta a}{a}\right)_2$	$(\Delta \eta)_2$
Ia.....	0.70	-----	0.45	22°	0.14	4.4°	0.08	1.2°
Ib.....	0.10	-3°	0.03	-2°	0.004	-0.05°	0.00	-0°
II.....	0.91	360°	0.29	85°	0.02	11.0°	0.004	3.4°
III.....	0.00	105°	0.00	22°	0.00	2.7°	0.00	0.8°

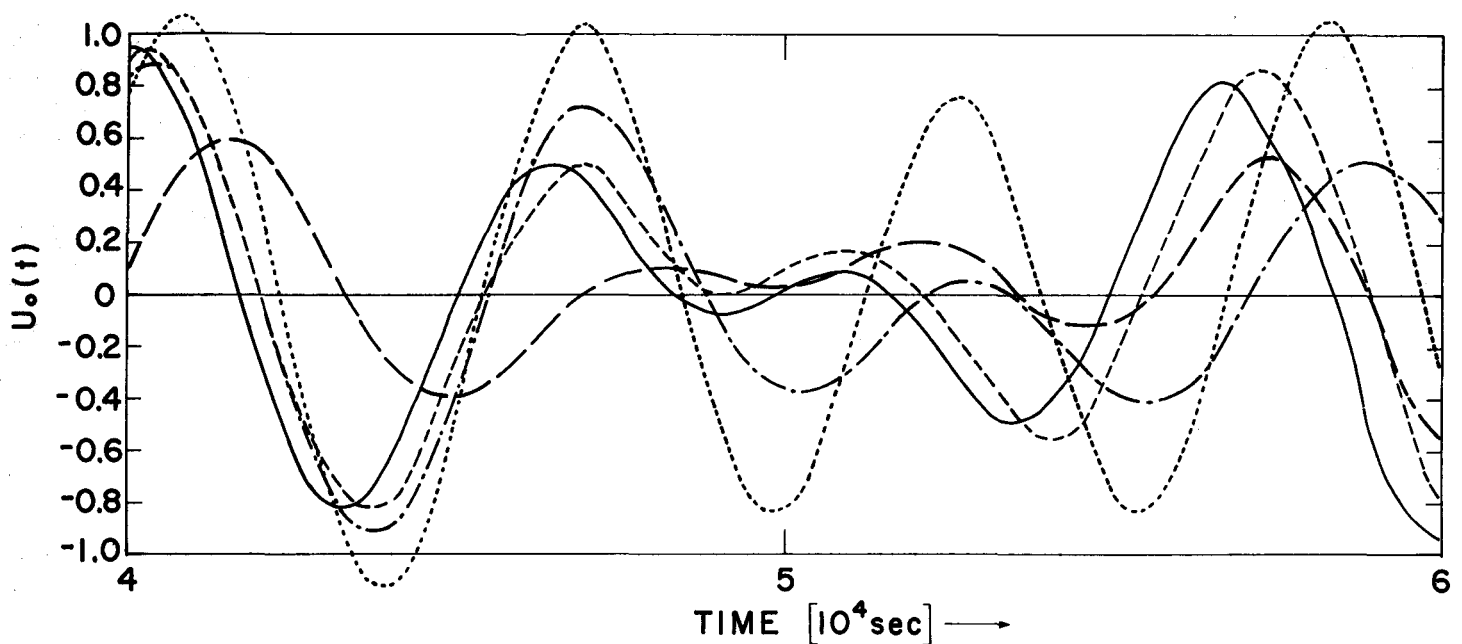


FIGURE 3.—Comparison between true and numerical solution $u_0(t)$ for $L=10\Delta x$. True solution (solid curve), scheme Ia (dotted), scheme Ib (dash-dot), scheme II (long dashes), scheme III (short dashes).

small artificial damping which was already predicted by the eigenvalues (8). The wave with phase velocity c_2 has very small errors in phase and possesses a numerical phase velocity which is larger than the true phase velocity; this is the only case where $|c^*| > |c|$. It should be mentioned that in the schemes Ia and Ib the errors showed a tendency to oscillate slightly with time. To obtain significant values, the errors in table 1 were interpolated linearly in time.

For $L=10\Delta x$, figure 3 shows the various solutions for u at $x=0$ as a function of t . The parameters have the same values as before. The true solution u is composed by a superposition of two waves with phase velocities c_1 and c_2 respectively. Applying our conditions, this gives

$$U(0, t) = \left(\cos \frac{2\pi}{L} U t \cos \frac{2\pi}{L} \gamma t \right) e^{\frac{-4\pi^2}{L^2} A t}.$$

We have selected in our example a time interval ranging from $t=4 \times 10^4$ sec. to $t=6 \times 10^4$ sec.

The amplification of the scheme Ia is not so obvious in this example, as the initial value of the amplitude $\frac{1}{2}(\alpha_1 + \alpha_2) = 1$ is only exceeded by about 10 percent. In another computation using a time step of $t=4 \times 10^2$ sec., as in the other schemes, values of more than double the initial amplitude appeared in this time interval. The solution of the Lax-Wendroff scheme is comparatively poor because the magnitude of the parameters $C_1^2=0.49$, $C_2^2=0.25$ implies relatively large amplitude and phase errors for the wavelength $L=10\Delta x$. The leapfrog scheme III gives by far the best approximation to the true solu-

tion. All schemes show, however, rapidly decreasing errors as the wavelength increases.

7. CONCLUSION

Our results lead to the conclusion that the leapfrog scheme is the most accurate one of all investigated in this paper. In nonlinear computations, however, the leapfrog approximation may develop the so-called nonlinear instability (Phillips [8]), stemming from the fact that energy transferred from longer to shorter waves is accumulated at short wavelengths in the order of $L=2\Delta x$. As the leapfrog scheme contains no artificial damping, this energy cannot be removed if there is no, or inadequate, frictional dissipation. As Richtmyer [12] pointed out, the artificial damping of short waves incorporated in the Lax-Wendroff scheme will probably prevent the accumulation of energy at short wavelengths and thus will give stable solutions also in the nonlinear sense.

Another argument is that in the leapfrog method there may appear computational modes different from the physical modes (Platzman [10]). In our linear equations these computational modes were excluded by a non-centered approximation of the initially unknown quantities $u_j^{(1/2)}$, $p_j^{(1/2)}$. Numerical integrations of the nonlinear primitive equations for a barotropic atmosphere performed by Phillips [9] with the leapfrog scheme revealed increasing truncation errors near boundaries; these errors were probably connected with the computational modes because they did not appear in a modified scheme which was essentially the two-step Lax-Wendroff scheme in two dimensions using Eliassen's [1] grid.

From these experiences it might be advisable to prefer the Lax-Wendroff scheme II for nonlinear equations. The smaller accuracy of this scheme can be compensated for by a refined net. Our results give some indication of the time and grid intervals needed to gain a certain accuracy for a certain wavelength.

On the other hand, however, it is also possible to force damping of short waves in the leapfrog scheme. This can, for instance, be accomplished by the common smoothing techniques, by selective filtering using Phillips' [8] method, or by applying appropriate dissipative terms. Referring to the latter, our "natural" second-order diffusion terms can be considered as such a means for damping waves. But corresponding higher order terms could also enforce damping, particularly of short waves. If, for example, terms proportional to the finite difference approximation of $\partial^{2m}u/\partial x^{2m}$ and $\partial^{2m}p/\partial x^{2m}$ ($m=1, 2, \dots$) were added to the u -equation and p -equation respectively, then the amplification factor (18) would contain an additional term proportional to ν^{2m} . With the proper sign and proper coefficients, this term implies a damping which is the more confined to short waves the larger m is. The influence on the phase lag in (20) would be in the order of $(\pi\Delta x/L)^{2m+1}$. Thus for sufficiently large m , these terms practically do not affect the longer waves, whereas short waves are eliminated. For $m>2$ this method is more effective in damping short waves than the Lax-Wendroff method. It has, however, the disadvantage that the order of the difference equations and thus the number of computational boundary conditions are raised.

Second-order linear or nonlinear diffusion terms, which are moreover physically justified, have been applied in practically all cases of numerical weather predictions using the leapfrog approximation to the primitive equations and they obviously did prevent the occurrence of nonlinear instability. Nonlinear diffusion terms were, for example, employed successfully by Smagorinsky [13] in his general circulation experiments. Also the artificial damping in the schemes Ib and II is basically produced by diffusion-like second order terms in the difference system with coefficients equal to $1/2U^2$ and $1/2(U^2+\gamma^2)$ respectively. This can be seen if the first steps in (Ib) and (II) are substituted into the second steps.

The difficulties connected with computational modes in the leapfrog scheme which were, as already mentioned, observed by Phillips [9] arose at a very special kind of boundary (near the overlapping boundaries of the Mercator and stereographic grids). Usually computational modes are suppressed by an appropriate initial approximation of the dependent variables at time $t=\frac{1}{2}\Delta t$ (Miyakoda [7]). Thus, in principle, there seems to be no hindrance to employing the leapfrog scheme also for nonlinear equations.

The scheme Ib is the most efficient one because it tends always to use the latest information and there-

fore occupies relatively small room in the computer. (The leapfrog scheme requires about twice as much storage room.) It also contains artificial damping which is proportional to $W^2\nu^4$ and thus depends on the particle velocity of the flow. Unfortunately it was not possible to get such an extensive insight into the behavior of this scheme as in the other cases. In our examples (table 1 and fig. 3) the scheme Ib was more accurate than the Lax-Wendroff scheme II, but less accurate than the leapfrog scheme III.

The scheme Ia is not to be recommended because its stability depends merely on the magnitude of the diffusion coefficient A and on the time step.

Although our examples are given for one space variable only, the principal features are maintained if the grid is extended to two space variables x and y . In this case, neglecting the diffusion term, the stability criterion for the schemes II and III should be changed to $(|U|+\gamma)^2 \left(\frac{\Delta t}{\Delta s}\right)^2 \leq 1/2$ where U is the scalar particle velocity and $\Delta s=\Delta x=\Delta y$, the grid interval. Stability for the scheme Ib is probably retained when $2\left(\frac{U\Delta t}{\Delta s}\right)^2 + \sqrt{2}\frac{\gamma\Delta t}{\Delta s} \leq 1$.

The extension of our staggered grid for the schemes II and III to two space variables can be done in essentially two different ways using either the already mentioned Eliassen [1] grid system, where the dependent variables are staggered in space too, or the more straightforward method proposed by Lilly [6], where all dependent variables are kept at the same grid point in the x,y -plane. There are also no difficulties in adopting the grid applied for the scheme Ib to two space variables.

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